

# A new 3D macroscopic model for shape memory alloys describing martensite reorientation

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## Abstract

In this paper we introduce a 3D phenomenological model for shape memory behavior, accounting for: martensite reorientation, asymmetric response of the material to tension/compression, different kinetics between forward and reverse phase transformation. We combine two modeling approaches using scalar and tensorial internal variables. Indeed, we use volume proportions of different configurations of the crystal lattice (austenite and two variants of martensite) as scalar internal variables and the preferred direction of stress-induced martensite as tensorial internal variable. Then, we derive evolution equations by a generalization of the principle of virtual powers, including microforces and micromovements responsible for phase transformation. In addition, we prescribe an evolution law for phase proportions ensuring different evolution laws during forward and reverse transformation of the oriented martensite.

**Key words:** shape memory, phase transformation, reorientation, internal variables

# 1 Introduction

In the last years shape memory alloys have been deeply investigated, from the point of view of modeling, analysis, and computation. Indeed, these materials present many important industrial applications (e.g. aeronautical, biomedical, structural, and earthquake engineering) due to their characteristic of superelasticity and shape memory effect.

It is known that the shape memory effect is the consequence of a (reversible) martensitic phase transformation between different configurations of the crystal lattice in the alloy: from a high symmetric phase, austenite, to a lower symmetric configuration, martensite. Austenite is a solid phase (present at high temperature) which can transform in martensite by means of a shearing mechanism. When transformation comes from thermal actions (lowering temperature) the result is a multi-direction martensite, in which variants compensate each other and there is no resulting macroscopic deformation. On the contrary, when transformation is obtained by loading, oriented martensite is formed in the stress direction, exhibiting a macroscopic deformation.

In particular, the research have been developed towards the aim of finding a flexible phenomenological model. Some reliable models have been proposed to predict the response of such materials. Among the others, we focus on two models developed in the framework of phase transitions.

The first, proposed by Frémond (cf., e.g., [7] and some generalizations [4], [5]) describes the behavior of shape memory in terms of (local) volume proportions of different configurations of the crystal lattice. More precisely, the austenite and two variants of martensite are taken into account. Note that the average behavior of different configurations is considered as the behavior of the equivalent single variant. The resulting model is able to describe phase transformation between different configurations. However, the model is obtained assuming that the direction of the transformation strain (associated to the detwinned martensite) is known.

The second model we are considering has been proposed in [10] and then generalized in [2]. In this case one internal variable describes the phenomenon: the transformation strain tensor. In this case, the internal variable describes the direction of martensite orientation. In addition, it leads to a simple and robust algorithm, based on plasticity-like return map. Thanks to this property the model has been used for implementation within finite element codes, allowing the simulation of complex SMA devices. However, some secondary effects are not included in this second model, as scalar and directional informations are tightly interconnected.

Thus, one could wonder how to get a deeper description of micro-phenomena,

possibly combining the main features of the two different approaches. Accordingly, the purpose of this paper is to answer to this idea combining the two theories describing secondary effects in the phase transitions as well as directional information for the transformation strain. Thus, both scalar and tensorial internal variables are introduced, accounting for the phase proportions (assuming that in each point the phases may coexist with different proportions) and for the orientation of the transformation strain associated to the detwinned martensite. We consider both proportion and direction as internal variables and we write evolution equations for both of them. We recall that an attempt in this direction has been performed in [1]. In particular, we prescribe an evolution law to capture asymmetric response of the material in tension-compression loading. In [3] a 1D model has been introduced to describe this kind of phenomenon, using an asymmetric energy depending on a tensorial variable. However, this model seems to be hard to be extended to higher dimensions. On the contrary, our approach (developing an asymmetric evolution theory for phase proportions) can apply to any space dimension.

## 2 The model

In this section we detail the derivation of the model. We mainly refer to the approach proposed by Frémond to describe the behavior of a thermomechanical system in terms of state and dissipative variables, as well as energy and dissipation functionals (see [7]). The main idea consists in assuming that (microscopic) phase transformations are due to micro-forces and micro-movements that have to be included in the global energy balance of the system (i.e. generalizing the principle of virtual powers). In particular, the equations governing the evolution of internal variables are recovered as balance equations (as for the momentum balance).

### 2.1 The state and dissipative variables

As it is known, phase transformations in the alloy are due to the phase transitions occurring in the microstructure configuration between austenite and twinned or detwinned martensite. In particular, detwinning manifests itself mainly through a shear strain so that we introduce a symmetric and deviatoric strain which appears in presence of the detwinned martensite. The (local) volume proportions of austenite and martensite variants is represented by phase parameters

$$\chi_A, \chi_M, \chi_S \in [0, 1], \quad \chi_A + \chi_M + \chi_S = 1. \quad (2.1) \quad \boxed{\text{varfase}}$$

More precisely,  $\chi_A$  stands for austenite,  $\chi_M$  for twinned martensite, and  $\chi_S$  for detwinned martensite. Furthermore,  $\mathbf{d}^{tr}$  is the direction of the deviatoric strain tensor associated to the detwinned martensite with  $\|\mathbf{d}^{tr}\| = \xi_s$  ( $\xi_s$  is the maximum amount for the detwinned martensite). Indeed, the deviatoric strain for detwinned martensite is given by  $\chi_S \mathbf{d}^{tr}$ . Then,  $\theta$  is the absolute temperature,  $\boldsymbol{\varepsilon}(\mathbf{u})$  the (symmetric) linearized strain tensor ( $\mathbf{u}$  is the vector of small displacements as we restrict ourselves to small deformations). Due to the internal constraint (2.1) on the phase proportions (coming by their physical meaning) we can restrict ourselves to consider just two independent phase variables  $(\chi_M, \chi_S)$  letting

$$\chi_A = 1 - \chi_M - \chi_S,$$

where

$$0 \leq \chi_M, \chi_S \leq 1, \quad \chi_M + \chi_S \leq 1.$$

Finally, let us use the notation  $\boldsymbol{\varepsilon}^e$  for the elastic component of the strain, so that it results

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \chi_S \mathbf{d}^{tr}.$$

Hence, the corresponding deviatoric strain  $\mathbf{e}$  is

$$\mathbf{e} := \boldsymbol{\varepsilon} - \frac{1}{3} \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I}$$

$\mathbf{I}$  being the identity matrix and  $\text{tr}(\cdot)$  the trace operator. If  $\boldsymbol{\sigma}$  is the Cauchy stress tensor, the deviatoric stress tensor  $\mathbf{S}$  is

$$\mathbf{S} := \boldsymbol{\sigma} - \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \mathbf{I} = \boldsymbol{\sigma} - \sigma_m \mathbf{I}.$$

As far as evolution, this is described by dissipative variables  $\chi_{Mt}$ ,  $\chi_{St}$ ,  $\mathbf{d}_t^{tr}$ , and  $\nabla \theta$ . These variables are in particular related to micro-velocities in the phase transformation.

*Remark 2.1.* Let us comment about the choice of state variables. The main idea consists in distinguishing between the norm and the direction of the inelastic strain. In this way, we are able to describe the presence of a product phase, to which a homogenized strain is associated, and a parent phase, in which we find only elastic strain. However, in the parent phase, we can also distinguish between the presence of twinned martensite and austenite. Thus, we get a more complex and free description of the phenomenon with respect to the Souza and Frémond models, which could be useful in some situations.

## 2.2 The energy and dissipation functionals

We introduce the following free energy functional (depending on state variables) as a combination of the energies associated to the single variants (combined with suitable proportions) and by an interaction energy, accounting also for internal constraints

$$\Psi(\boldsymbol{\varepsilon}, \mathbf{d}^{tr}, \chi_M, \chi_S, \theta) = \Psi_{el} + \Psi_{id} + \Psi_{ch} + \Psi_v \quad (2.2)$$

where

$$\begin{aligned} \Psi_{el} &= \left( \frac{\lambda}{2} + \frac{\mu}{3} \right) (\text{tr } \boldsymbol{\varepsilon})^2 + \mu \|\mathbf{e} - \chi_S \mathbf{d}^{tr}\|^2 \\ \Psi_{id} &= c_s((\theta - \theta_0) - \theta \log \theta) \\ \Psi_{ch} &= (1 - \chi_M - \chi_S)h_A(\theta) + \chi_M h_M(\theta) + \chi_S h_S(\theta) + \mathbf{h}_d(\theta) : \mathbf{d}^{tr} \\ \Psi_v &= I_K(\chi_M, \chi_S) + I_{\xi_s}(\|\mathbf{d}^{tr}\|) + \Psi_{int}(\chi_M, \chi_S). \end{aligned} \quad (2.3)$$

Here  $c_s > 0$  is the specific heat,  $\lambda$  and  $\mu$  are the Lamé constants;  $h_A, h_S, h_M, \mathbf{h}_d$  are smooth thermal functions whose regularity will be specified later on (to ensure compatibility with thermodynamics). The function  $I_K$  is the indicator function of the convex set  $K$

$$K := \{(\chi_M, \chi_S) \in \mathbf{R}^2 : 0 \leq \chi_M, \chi_S \leq 1, \chi_M + \chi_S \leq 1\},$$

i.e. it is  $I_K(\chi_M, \chi_S) = 0$  if  $(\chi_M, \chi_S) \in K$ , while  $I_K(\chi_M, \chi_S) = +\infty$  otherwise. The function  $I_{\xi_s}$  forces  $\|\mathbf{d}^{tr}\| = \xi_s$ . Indeed, it is  $I_{\xi_s}(\|\mathbf{d}^{tr}\|) = 0$  if  $\|\mathbf{d}^{tr}\| = \xi_s$  and it is  $+\infty$  otherwise.  $\Psi_{int}$  is a (sufficiently) smooth function accounting for interaction energy. As a possible choice for the interaction energy  $\Psi_{int}$ , we could simply consider

$$\begin{aligned} \Psi_{int}(\chi_M, \chi_S) &= C_{MS}\chi_M\chi_S + (C_{AM}\chi_M + C_{AS}\chi_S)(1 - \chi_M - \chi_S) \\ &\quad + C_{AMS}\chi_M\chi_S(1 - \chi_M - \chi_S), \end{aligned} \quad (2.4)$$

where  $C_{MS}, C_{AM}, C_{AS}, C_{AMS}$  are positive constants.

*Remark 2.2.* Note that for  $(\chi_M, \chi_S)$  we have introduced a convex constraint forcing  $(\chi_M, \chi_S) \in K$ . The constraint on  $\mathbf{d}^{tr}$  is convex w.r.t. to its norm as it is  $\|\mathbf{d}^{tr}\| = \xi_s$ .

Now, let us introduce the pseudo-potential of dissipation, which is a positive convex functional depending on dissipative variables, vanishing for vanishing dissipation (cf. [9]). We have

$$\phi(\chi_{Mt}, \chi_{St}, \mathbf{d}_t^{tr}, \nabla \theta) = |\chi_{Mt}| + \phi_S(\chi_S, \boldsymbol{\sigma}, \chi_{St}) + \chi_S \|\mathbf{d}_t^{tr}\| + \frac{1}{2\theta} |\nabla \theta|^2. \quad (2.5) \quad \boxed{\text{pseudo}}$$

Note that,  $\phi$  is considered to possibly ensure an asymmetric behavior in tension and compression and for forward and backward transformation. This is due to the choice of the function  $\phi_S$  (noting that it possibly depends on the stress and  $\chi_S$ ). Indeed, this choice generalizes the classical situation for rate-independent systems, where it is

$$\phi_S(\chi_S, \boldsymbol{\sigma}, \chi_{St}) = |\chi_{St}|. \quad (2.6)$$

Actually,  $\phi_S$  is required to be rate independent with respect to  $\chi_{St}$ . Hence, accounting for a possible dependence in the evolution on the stress (e.g., for tension-compression behavior) and on the volume of already detwinned martensite, we get as a further possible example

$$\phi_S(\chi_S, \boldsymbol{\sigma}, \chi_{St}) = d(\chi_S, \boldsymbol{\sigma})(\chi_{St})^+ + |\chi_{St}| \quad (2.7) \quad \boxed{\text{specifico}}$$

where  $(f)^+ = f$  if  $f \geq 0$  and  $(f)^+ = 0$  if  $f \leq 0$  and  $d$  is a sufficiently smooth function. From now on we deal in particular with (2.7).

asim *Remark 2.3.* Note that we could refine the model, e.g. adding in (2.5) a term as  $\widehat{d}(\chi_S, \boldsymbol{\sigma})(\chi_{St})^-$  for decreasing evolution of the product phase.

## 2.3 The equations

We consider a smooth bounded domain  $\Omega \subseteq \mathbf{R}^3$  with  $\Gamma = \partial\Omega$  split into  $\Gamma_1 \cup \Gamma_2$  (with  $\Gamma_i$  disjoint subset,  $\Gamma_1$  with strictly positive measure).

We assume that a generalized version of the principle of virtual powers holds, accounting for internal microforces responsible for phase transitions. Thus, the first principle of thermodynamics reads as follows

$$e_t + \operatorname{div} \mathbf{q} = r + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t + B_M \chi_{Mt} + B_S \chi_{St} + \mathbf{B} : \mathbf{d}_t^{tr} \quad \text{in } \Omega \quad (2.8) \quad \boxed{\text{enI}}$$

the right hand side being the power of interior forces and the heat source  $r$ . Here,  $e$  is the internal energy,  $\mathbf{q}$  the heat flux,  $(B_M, B_S)$  and  $\mathbf{B}$  internal (microscopic) forces responsible for the phase transformation (i.e. the evolution of internal variables). The heat flux satisfies boundary condition ( $h$  is a known flux through the boundary)

$$\mathbf{q} \cdot \mathbf{n} = h \quad \text{on } \Gamma. \quad (2.9)$$

Hence, by the principle of virtual powers we get the quasi-static momentum balance

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega \quad (2.10)$$

with boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \quad (2.11) \quad \boxed{\text{boundu1}}$$

$$\boldsymbol{\sigma} \mathbf{n} = \mathbf{t} \quad \text{on } \Gamma_2 \quad (2.12) \quad \boxed{\text{boundu2}}$$

$\mathbf{f}$  being a volume force, while  $\mathbf{t}$  is a traction applied on a part of the boundary.

Analogously, the evolution of the phases depends on internal forces which are included in the energy balance of the system. Thus, we get two balance equations, one for the evolution of the phase proportions (related to  $(B_M, B_S)$ ) and one for the evolution of the tensor  $\mathbf{d}^{tr}$  (related to  $\mathbf{B}$ ), i.e.

$$(B_M, B_S) = (0, 0) \quad \text{in } \Omega \quad (2.13)$$

$$\mathbf{B} = \mathbf{0} \quad \text{in } \Omega. \quad (2.14)$$

## 2.4 The constitutive relations

We need to prescribe constitutive relations for the involved physical quantities. The internal energy is

$$e = \Psi - \theta \eta$$

where the entropy  $\eta$  is prescribed by

$$\begin{aligned} \eta = -\frac{\partial \Psi}{\partial \theta} &= c_s \log \theta - h'_A(\theta)(1 - \chi_M - \chi_S) \\ &- h'_M(\theta)\chi_M - h'_S(\theta)\chi_S - \mathbf{h}'_d(\theta) : \mathbf{d}^{tr}. \end{aligned} \quad (2.15)$$

The Cauchy stress tensor is

$$\boldsymbol{\sigma} = \mathbf{S} + \sigma_m \mathbf{I},$$

with

$$\sigma_m = \frac{\partial \Psi}{\partial \text{tr } \boldsymbol{\varepsilon}} = \left( \lambda + \frac{2}{3} \mu \right) \text{tr } \boldsymbol{\varepsilon},$$

and

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{e}} = 2\mu(\mathbf{e} - \chi_S \mathbf{d}^{tr}). \quad (2.16) \quad \boxed{\text{Sgrande}}$$

Hence, we get

$$(B_M, B_S) = (B_M^{nd}, B_S^{nd}) + (B_M^d, B_S^d) = \frac{\partial \Psi}{\partial (\chi_M, \chi_S)} + \frac{\partial \phi}{\partial (\chi_{Mt}, \chi_{St})}. \quad (2.17)$$

More precisely, letting

$$s(x) = \frac{x}{|x|} \text{ if } x \neq 0, \quad s(0) = [-1, 1],$$

and

$$H(x) = 1 \text{ if } x > 0, \quad H(x) = 0 \text{ if } x < 0, \quad H(0) = [0, 1]$$

there holds

$$\begin{aligned} B_M^{nd} &= -h_A(\theta) + h_M(\theta) + \frac{\partial \Psi_{int}}{\partial \chi_M} + \gamma_M, \\ B_M^d &= s(\chi_{Mt}), \end{aligned} \tag{2.18} \quad \boxed{\text{sceIgoB1}}$$

and (choosing  $\phi_S$  as in (2.7))

$$\begin{aligned} B_S^{nd} &= -h_A(\theta) + h_S(\theta) + \frac{\partial \Psi_{int}}{\partial \chi_S} - 2\mu(\mathbf{e} - \chi_S \mathbf{d}^{tr}) : \mathbf{d}^{tr} + \gamma_S, \\ B_S^d &= \frac{\partial \phi_S}{\partial \chi_{St}} = s(\chi_{St}) + d(\chi_S, \boldsymbol{\sigma}) H(\chi_{St}) \end{aligned} \tag{2.19} \quad \boxed{\text{sceIgoB2}}$$

with

$$(\gamma_M, \gamma_S) \in \partial I_K(\chi_M, \chi_S). \tag{2.20}$$

Finally, we consider

$$\mathbf{B} = \mathbf{B}^{nd} + \mathbf{B}^d = \frac{\partial \Psi}{\partial \mathbf{d}^{tr}} + \frac{\partial \phi}{\partial \mathbf{d}_t^{tr}} \tag{2.21}$$

where

$$\begin{aligned} \mathbf{B}^{nd} &= -2\mu\chi_S(\mathbf{e} - \chi_S \mathbf{d}^{tr}) + \mathbf{h}_d(\theta) + \gamma \mathbf{d}^{tr} \\ \mathbf{B}^d &= \chi_S \mathbf{s}(\mathbf{d}_t^{tr}) \end{aligned} \tag{2.22}$$

using the notation

$$\mathbf{s}(\mathbf{d}_t^{tr}) = \frac{\mathbf{d}_t^{tr}}{\|\mathbf{d}_t^{tr}\|} \text{ if } \mathbf{d}_t^{tr} \neq 0, \quad \mathbf{s}(\mathbf{0}) = \{\mathbf{w} : \|\mathbf{w}\| \leq 1\}.$$

and letting

$$\gamma \in \frac{1}{\xi_s} \partial I_{\xi_s}(\|\mathbf{d}^{tr}\|) = \partial I_{\xi_s}(\|\mathbf{d}^{tr}\|).$$

As far as the heat flux, we assume (Fourier law)

$$\mathbf{q} = -\theta \frac{\partial \Phi}{\partial \nabla \theta} = -\nabla \theta. \tag{2.23}$$



### 3 The PDE system

#### 3.1 The first principle

Combining constitutive relations with the balance laws, we get the PDE system we deal with. First let us discuss the energy balance, from which we show that the model is thermodynamically consistent. The equation governing the evolution of the temperature is recovered from (2.8). After using the chain rule and by the constitutive relations, we get

$$\theta(\eta_t + \operatorname{div} \frac{\mathbf{q}}{\theta}) - r = \frac{\partial \Phi}{\partial(\chi_{Mt}, \chi_{St})} \cdot (\chi_{Mt}, \chi_{St}) + \frac{\partial \Phi}{\partial \mathbf{d}_t^{tr}} : \mathbf{d}_t^{tr} + \frac{\partial \Phi}{\partial \nabla \theta} \cdot \nabla \theta \geq 0 \quad (3.24)$$

from which the second principle of thermodynamics follows, once  $\theta > 0$  (it is the absolute temperature). Note in particular that we have strongly exploited the fact that  $\partial \Phi$  turns out to be a maximal monotone operator with  $\mathbf{0} \in \partial \Phi(\mathbf{0})$ . The resulting equation is

$$\begin{aligned} & \theta_t(c_s - \theta(h_M''(\theta)\chi_M + h_S''(\theta)\chi_S + h_A''(\theta)(1 - \chi_M - \chi_S) + \mathbf{h}_d''(\theta) : \mathbf{d}^{tr})) \\ & + \theta h_A'(\theta)(\chi_M + \chi_S)_t - \theta h_M'(\theta)\chi_{Mt} - \theta h_S'(\theta)\chi_{St} - \theta \mathbf{h}_d'(\theta) : \mathbf{d}_t^{tr} - \Delta \theta \\ & = |\chi_{Mt}| + |\chi_{St}| + d(\chi_S, \boldsymbol{\sigma})(\chi_{St})^+ |\chi_{St}| + \chi_S \|\mathbf{d}_t^{tr}\|. \end{aligned} \quad (3.25)$$

In particular, we have to assume that  $h_A, h_M, h_S, \mathbf{h}_d$  are smooth functions such that

$$(c_s - \theta(h_M''(\theta)\chi_M + h_S''(\theta)\chi_S + h_A''(\theta)(1 - \chi_M - \chi_S) + \mathbf{h}_d''(\theta) : \mathbf{d}^{tr})) \geq C > 0.$$

#### 3.2 The evolution

Combining constitutive relations with momentum balance, it follows

$$-\operatorname{div} \left( (\lambda + \frac{2}{3}\mu) \operatorname{tr} \boldsymbol{\varepsilon} \mathbf{I} + 2\mu(\mathbf{e} - \chi_S \mathbf{d}^{tr}) \right) = \mathbf{f}, \quad (3.26)$$

combined with (2.11), (2.12). Then, by definition of  $B_M$  and  $B_S$ , the evolution equations for  $(\chi_{Mt}, \chi_{St})$  are written as

$$s(\chi_{Mt}) + (h_M(\theta) - h_A(\theta)) + \frac{\partial \Psi_{int}}{\partial \chi_M} + \gamma_M = 0 \quad (3.27) \quad \boxed{\text{ev1phase}}$$

and

$$s(\chi_{St}) + d(\chi_S, \boldsymbol{\sigma})H(\chi_{St}) + (h_S(\theta) - h_A(\theta)) - 2\mu(\mathbf{e} - \chi_S \mathbf{d}^{tr}) : \mathbf{d}^{tr} + \gamma_S = 0 \quad (3.28) \quad \boxed{\text{ev2phase}}$$

where

$$(\gamma_M, \gamma_S) \in \partial I_K(\chi_M, \chi_S).$$

Note that  $\partial I_K(\chi_M, \chi_S) = (0, 0)$  if  $(\chi_M, \chi_S)$  belongs to the interior of  $K$ , while it is given by the normal cone to the boundary if  $(\chi_M, \chi_S) \in \partial K$ .

Finally, the evolution equation for  $\mathbf{d}^{tr}$  is given by

$$\begin{aligned} \chi_S \mathbf{s}(\mathbf{d}_t^{tr}) - 2\mu \chi_S (\mathbf{e} - \chi_S \mathbf{d}^{tr}) + \mathbf{h}_d(\theta) \\ + \frac{\partial \Psi_{int}}{\partial \chi_S} + \gamma \mathbf{d}^{tr} = \mathbf{0}, \quad \gamma \in \partial I_{\xi_S}(\|\mathbf{d}^{tr}\|). \end{aligned} \quad (3.29) \quad \boxed{\text{ev3phase}}$$

*Remark 3.1.* Note that the coefficient  $\chi_S$  of the evolution term  $\mathbf{s}(\mathbf{d}_t^{tr})$  ensures that in the absence of detwinned martensite there is not dissipative contributions involving  $\mathbf{d}^{tr}$ .

### 3.3 An equivalent formulation

Let us now introduce ( $B_M$  and  $B_S$  are defined as in (2.18) and (2.19))

$$F_M(B_M^{nd}) = |B_M^{nd}| - 1 \quad (3.30)$$

$$F_S(B_S^{nd}) = |B_S^{nd}| - R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) \quad (3.31)$$

where

$$\begin{aligned} R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) &= 1 \text{ if } B_S^{nd} < 0 \\ \text{and } R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) &= 1 + d(\chi_S, \boldsymbol{\sigma}) \text{ if } B_S^{nd} \geq 0. \end{aligned}$$

Then, we can rewrite the evolution of the phases (3.27) and (3.28) as follows

$$\chi_{Mt} = \zeta_M \frac{B_M^{nd}}{|B_M^{nd}|}, \quad (3.32)$$

$$\chi_{St} = \zeta_S \frac{B_S^{nd}}{|B_S^{nd}|}, \quad (3.33)$$

$$\zeta_i F_i = 0 \quad i = M, S, \quad (3.34)$$

$$F_i(B_i^{nd}) \leq 0, \quad i = M, S. \quad (3.35)$$

Note that  $F_i$  play the role of yield functions (see, e.g., [8]).

Analogously we may introduce

$$F_d = \|\mathbf{B}^{nd}\| - \chi_S$$

letting

$$\mathbf{d}_t^{tr} = \zeta_d \frac{\mathbf{B}^{nd}}{\|\mathbf{B}^{nd}\|},$$

with

$$\zeta_d F_d = 0, \quad F_d \leq 0.$$

## 4 Some examples

In the following we explore the model performances limiting the discussion only to the case of a proportional loading state, i.e., neglecting the reorientation process. Accordingly, to simplify the discussion, we set  $\mathbf{h}_d(\theta) = \mathbf{0}$ ,  $\mathbf{d}^{tr}$  in the same direction of  $\mathbf{S}$  (and  $\mathbf{e}$ ), assuming also  $\mathbf{d}_t^{tr} = \mathbf{0}$ . Under these simplifying positions, we may set (see (2.16))

$$\frac{\mathbf{B}^{nd}}{\|\mathbf{B}^{nd}\|} = \frac{\mathbf{e}}{\|\mathbf{e}\|} = \frac{\mathbf{d}^{tr}}{\|\mathbf{d}^{tr}\|}, \quad (4.36)$$

and

$$\|\mathbf{S}\| = 2\mu(\|\mathbf{e}\| - \chi_S). \quad (4.37)$$

Moreover, we distinguish between two different possible situations, one in which we consider only evolution of the stress-induced martensite and one in which we consider only evolution of the temperature-induced martensite, as discussed in the following. For both problems we start from a material completely in austenite (i.e.,  $\chi_S = \chi_M = 0$ ).

### 4.1 Case 1: temperature-induced effect

For this problem we assume to start from  $\sigma = 0$  and to vary only the temperature. Accordingly, only a variation of  $\chi_M$  can be produced.

The problem is governed by the following set of equations (see (2.18)):

$$\begin{cases} B_M^{nd} = -h_A(\theta) + h_M(\theta) + \frac{\partial \Psi_{int}}{\partial \chi_M} + \gamma_M, \\ F_M(B_M^{nd}) = |B_M^{nd}| - 1 \\ \chi_{Mt} = \zeta_M \frac{B_M^{nd}}{|B_M^{nd}|}, \\ \zeta_M F_M = 0 \quad F_M(B_M^{nd}) \leq 0. \end{cases} \quad (4.38)$$

We assume to first properly cool and then heat the material (see Figure ??).

In Figures ??-?? we report the evolution of the thermodynamic force  $B_M^{nd}$  versus the temperature  $\theta$ , of the temperature-induced martensite  $\chi_S$  versus the temperature  $\theta$ , of the quantity  $\gamma_M$  versus the the temperature  $\theta$ .

It can be observed that during cooling the model is able to reproduce a process in which the multi-variant martensite is produced and then during heating a process in which the multi-variant martensite is progressively extinguished. The forward and reverse phase transformations are perfectly symmetric.

## 4.2 Case 2: stress-induced effect

For this problem we assume to start from  $\sigma = 0$  and to vary only the stress. Accordingly, only a variation of  $\chi_S$  can be produced.

The problem is governed by the following set of equations ((2.19)):

$$\begin{cases} B_S^{nd} = -h_A(\theta) + h_S(\theta) + \frac{\partial \Psi_{int}}{\partial \chi_S} - 2\mu(\|\mathbf{e}\| - \chi_S) + \gamma_S, \\ F_S(B_S^{nd}) = |B_S^{nd}| - R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) \\ \chi_{St} = \zeta_S \frac{B_S^{nd}}{|B_S^{nd}|}, \\ \zeta_S F_S = 0 \quad F_S(B_S^{nd}) \leq 0. \end{cases} \quad (4.39)$$

where (letting (2.7) holds)

$$\begin{aligned} R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) &= 1 \text{ if } B_S^{nd} < 0 \text{ and} \\ R(B_S^{nd}, \chi_S, \boldsymbol{\sigma}) &= 1 + d(\chi_S, \boldsymbol{\sigma}) \text{ if } B_S^{nd} \geq 0. \end{aligned} \quad (4.40)$$

We assume to first properly load and then unload the material (see Figure ??).

In Figures ??-?? we report the evolution of the thermodynamic force  $B_S^{nd}$  versus the applied stress  $\sigma$ , of the stress-induced martensite  $\chi_S$  versus the applied stress  $\sigma$ , of the quantity  $\gamma_S$  versus the applied stress  $\sigma$ , of the applied stress  $\sigma$  versus the strain  $\epsilon$ .

It can be observed that during loading the model is able to reproduce a process in which the single-variant martensite is produced and then during unloading a process in which the single-variant martensite is progressively extinguished. The forward and reverse phase transformation are unsymmetric.

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